Optimal Range for the iid Test Based on Integration across the Correlation Integral

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ABSTRACT

This paper builds on Kočenda (2001) and extends it in three ways. First, new intervals of the proximity parameter ε (over which the correlation integral is calculated) are specified. For these ε-ranges new critical values for various lengths of the data sets are introduced and through Monte Carlo studies it is shown that within new ε-ranges the test is even more powerful than within the original ε-range. The range that maximizes the power of the test is suggested as the optimal range. Second, an extensive comparison with existing results of the controlled competition of Barnett et al. (1997) as well as broad power tests on various nonlinear and chaotic data is provided. Test performance with real (exchange rate) data is provided as well. The results of the comparison strongly favor our robust procedure and confirm the ability of the test in finding nonlinear dependencies as well its function as a specification test. Finally, new user-friendly and fast software is introduced.

Key Words: chaos, nonlinear dynamics, correlation integral, Monte Carlo, single-blind competition, power tests, high-frequency economic and financial data

JEL Classification: C14, C15, C52, C87, F31, G12.

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1. INTRODUCTION

Growing interest as well as practical needs of researchers in testing for nonlinearity and chaos in high-frequency economic and financial data has prompted the design of various methods to accomplish the task. Among them, a well known BDS test was devised by Brock, Dechert, Scheinkman and LeBaron (1996) as a non-parametric method of testing for nonlinear patterns in time series.[1] The method is based on the correlation integral described by Grassberger and Procaccia (1983) and is unique in its ability to detect nonlinearities independent of linear dependencies in the data. The null hypothesis is that data in a time series are independently and identically distributed (iid); an alternative is not specified. In order to conduct the BDS test, two free variables (embedding dimension m and proximity parameter ε) must be chosen ex ante, with limited guidance from statistical theory; thus it is likely that inappropriate values may be chosen. Kočenda (2001) has suggested an alternative test which, through integrating across the correlation integral, avoids arbitrary selection of the proximity parameter ε and allows for running the test across an empirically endorsed set of embedding dimensions m.

This paper builds on Kočenda (2001) and increases the operational ability of the alternative test in three ways. First, we improve the choice of the interval of the proximity parameter ε over which the correlation integral is calculated and suggest the optimal range that maximizes the power of the test. We also bring necessary sets of critical values for various lengths of data and provide an analysis of their sensitivity with respect to the choice of proximity parameter range. Second, we compare the test with existing results of the single-blind controlled competition of Barnett et al. (1997) and in addition perform power tests on various chaotic (nonlinear) and noisy chaotic data. The results strongly attest to our robust procedure. The performance of the test surpasses that of other tests for non-linear dependencies. The improvement allows for new economic insights especially when the procedure is used on financial data as a specification test. Third, new compact software to run the test, as well as allowing for associated simulations, is introduced as freeware.

Section 2 provides a brief theoretical background and describes the motivation for ε-range choice in greater detail. Section 3 deals with the choice of ε-range and supplies critical values generated by the Monte Carlo technique. Section 4 examines power tests, brings forth an analysis of the ε-range when this is expanded and contracted, defines the optimal ε-range (that maximizes the power of the test), tabulates appropriate critical values, and contains a sensitivity analysis of critical values over different ranges of proximity parameter ε. Section 5 provides analysis of the test performance with the real (exchange rate) data, and Section 6 briefly concludes.

[1] We cite other well known tests later in Section 4.1.

2. THEORETICAL BACKGROUND AND MOTIVATION

Chaotic systems of low dimensionality can generate seemingly random numbers that may give an impression of white noise, thereby hiding their true nature. Under presumed randomness, a nonlinear pattern can hide without being detected. Exchange rates, stock market returns and other macroeconomic variables of generally high frequency, for example, may originate from low-dimensional chaos. Detection of nonlinear *hidden patterns* in such time series provides important information about their behavior and improves forecasting ability over short time periods.[2]

The analysis of chaotic systems often starts with computing a correlation dimension because of the ease of computation and the availability of sampling theory. The aforementioned BDS test is based on such a technique and was designed to detect hidden patterns in stochastic time series. This test is a non-parametric test of the null hypothesis that the data are independently and identically distributed (iid) against an unspecified alternative. The procedure has power against both deterministic and stochastic systems. The BDS test is a well-known standard procedure; it is widely used for its ability to deal with stochastic time series, which makes its application in modern macroeconomics and financial economics extremely appealing.

While the BDS statistic is easy to compute, it suffers from an obvious drawback - the values of two parameters, proximity parameter ε (also referred to as tolerance distance or metric bound) and embedding dimension m, must be determined *ex ante*.[3] Further, the BDS statistic, when used for testing, has often been evaluated for only few values of the proximity parameter ε. This was brought about, in part, by the Monte Carlo studies of Hsieh and LeBaron (1988) who tested the asymptotic normality of the statistic for three values of the parameter, and tabulated the corresponding critical values.

The alternative test of Kočenda (2001) suggests considering an OLS-estimate of the correlation dimension (defined presently) over a range of ε-values, and is thus closer in spirit to the original correlation dimension than is the BDS test (for full details see the

[2] Recent advances in research of chaos allow researchers to control chaotically behaving systems in various fields of physics, biology, chemistry, and medicine. Effective control for chaos in economics does not seem to be more realistic than discovering Shangri-la, though.

[3] Some guidance can be found for example in Dechert (1994), Brock, Dechert, Scheinkman and LeBaron

original paper).[4] The test rests upon the concept of the correlation integral, developed by Grassberger and Procaccia (1983). Formally, let $\{x_t\}$ be a scalar time series of the size T generated randomly according to a density function f. Form m-dimensional vectors, called m-histories, $x_t^m = (x_t, x_{t+1}, \ldots, x_{t+m-1})$. The sample correlation integral (or correlation sum) at embedding dimension m is computed as

$$C_{m,T}(\varepsilon) = 2\sum_{t=1}^{T_{m-1}} \sum_{s=t+1}^{T_m} I_\varepsilon(x_t^m, x_s^m)/(T_m(T_m - 1)), \tag{2.1a}$$

where $T_m = T - m + 1$, and $I_\varepsilon(x_t^m, x_s^m)$ is an indicator function of the event $\left\| x_t^m - x_s^m \right\| = \max_{i=0,1,\ldots m-1} \left| x_{t+i} - x_{s+i} \right| < \varepsilon$. Further, the correlation integral at embedding dimension m is defined as

$$C_m(\varepsilon) = \lim_{T\to\infty} C_{m,T}(\varepsilon), \tag{2.1b}$$

Thus, the sample correlation integral measures the fraction of pairs that lie within the tolerance distance ε for the particular embedding dimension m. If, in the limit of large T and small ε, the correlation integral scales as ε^D, then the exponent D defines the correlation dimension as

$$D = \lim_{\varepsilon\to 0}\lim_{T\to\infty} \frac{\ln C_{m,T}(\varepsilon)}{\ln \varepsilon}. \tag{2.2}$$

The alternative statistic uses a number of tolerance distances chosen from a specific range for each particular embedding dimension by calculating the slope of the log of the correlation integral versus the log of the proximity parameter over a broad range of values of the proximity parameter. The estimates of the correlation dimension, slope coefficients β_m, can be estimated as

$$\beta_m = \frac{\sum_\varepsilon \left(\ln(\varepsilon) - \overline{\ln(\varepsilon)}\right)\cdot\left(\ln(C_{m,T}(\varepsilon)) - \overline{\ln(C_{m,T}(\varepsilon))}\right)}{\sum_\varepsilon \left(\ln(\varepsilon) - \overline{\ln(\varepsilon)}\right)^2}, \tag{2.3a}$$

which equals to calculating the slope coefficient β_m from the least squares regression

$$\ln\left(C_{m,T}(\varepsilon_i)\right) = \alpha_m + \beta_m \ln(\varepsilon_i) + u_i; \quad i = 1, \ldots, n \tag{2.3b}$$

where $\ln(\varepsilon)$ is the logarithm of proximity parameter (tolerance distance), $\ln(C_{m,T}(\varepsilon))$ is

(1996), de Lima (1992), and Hsieh and LeBaron (1988).

[4] It is worthwhile noting that originally an important reason to develop the BDS test was that point estimates

the logarithm of the sample correlation integral (correlation sum), m is the embedding dimension, and the variables with a bar denote the mean of their counterparts without a bar.[5]

Since a range of different tolerance distances ε is used the slope coefficients β_m do not depend on an arbitrary choice of ε. When computing the slope coefficient estimates of β_m, a cut-off point was set to eliminate the erratic portion of the trajectories at the highest embedding dimensions, m (7-10). The use of cut-off point effectively guarantees that the curve constructed from successive correlation dimensions over a given ε-range contains only a linear part and the erratic part is left out. Further, this approach means that the proximity parameters ε (used when particular estimates of the β_m coefficients are calculated) are in the range where the "scaling property" holds (e.g. $C_m(\varepsilon) \sim \varepsilon^{\alpha}$ for some α; thus proximity parameters ε are in the "scaling region" (see Theiler and Lookman, 1993, and Diks, 2004, for details). Such a cut-off point does not affect the analysis for lower embedding dimensions m, but considerably reduces the increasing variance as embedding dimension m grows larger and tolerance distance ε becomes smaller ($\varepsilon \rightarrow 0$). The cut-off point represents the number of matches that maximizes the power of the test or, implicitly, minimizes error of the second kind.[6]

As for the choice of embedding dimension m, a range of empirically endorsed dimensions m is used, which gives enough variety to capture a more complex dimensional structure without eliminating unexplored opportunities. One theoretical feature of the slope coefficients β_m is that under the null hypothesis that the data are iid, these slopes should equal the respective embedding dimension m at which the statistic is calculated (i.e. $\beta_m = m$).[7] However, slope coefficient estimates β_m are smaller than respective embedding dimension m, i.e. $\beta_m \leq m$. For details see Kočenda (2001).

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of the correlation dimension were very unstable across values of ε.

[5] As β_m is, in fact, an OLS estimate of the slope coefficient, by econometric tradition it should be labeled as $\hat{\beta}_m$. For the sake of notational simplicity, we decided to omit the hat.

[6] By simulation it was found that such a number lies in the interval between 40 to 50. To be on the safe side, the value of the correlation integral was constrained to be 50. The "cut-off" value for $C_m(\varepsilon)$ must be chosen before slope coefficient estimates are computed. $C_m(\varepsilon) = 50$ resulted from simulations that were compared with various trajectories resulting from the analysis conducted on different time series.

[7] See Hsieh (1991) for details.

3. PROXIMITY PARAMETER RANGE AND CRITICAL VALUES

Kočenda (2001) performed a Monte Carlo study with 10,000 replications of the distribution of the β_m statistic under the null hypothesis of iid data.[8] Critical values were tabulated for data-length of 500, 1000, and 2500 observations allowing for nine embedding dimensions m (2-10). The range of proximity parameter ε, for which the critical values were generated, extends over the specific interval: n = 41 proximity parameters ε ranging over the interval (0.25σ, 1.00σ) in proportionally equal increments (σ being standard deviation of a sample).

The above original interval is chosen sensibly to allow for hidden patterns corresponding to very narrow tolerance distances. However, a single ε-range prevents observing whether and how sensitive the tabulated critical values are to a choice of different ε-range. Further, the originally chosen ε-range does not need to be an optimal range; e.g. the range that, when used, maximizes the power of the test. We elaborate on this issue in the following sections.

3.1 Range Selection

The issue of the ε-range choice is complicated by the fact that we cannot theoretically derive a correct range of proximity parameters. This is due to the fact that the behavior of the β_m statistic within an ε-range is closely related to the composition of the analyzed data. Therefore it is possible that one ε-range is more appropriate for some kind of data and a different ε-range for another one. (We will come back to these issues in Section 4.5) For this reason, we select two additional ranges of proximity parameters to study whether and how the critical values behave.[9] We proceed with calculating critical values, then compare them for three different intervals, and finally perform a series of power tests to expose the range that provides the test with the greatest power against the null hypothesis among the three given ranges. Since both the BDS and the Kočenda tests start with

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[8] A compound random number generator based on the idea of Collings (1987) and constructed from 17 generators described by Fishman and Moore (1982) was employed to generate iid data.

[9] The issue of different ε-ranges is discussed also in Belaire-Franch (2003) who argues that although the power of Kočenda's test can be more powerful than the BDS test, more than one ε-range should be used. The two additional ranges used in his study were constructed only as an additive extension to the original

computing the correlation integral, we use the BDS literature as a point of reference on what proximity parameters are often used as required entries to compute the test. Table 1 offers a summary of selected representative literature dealing with the issue.

To repeat, Kočenda (2001) provides critical values for proximity parameter in a range of (0.25σ, 1.00σ); this is our first (original) interval. This range is the most discriminating range of the proximity parameters ε, which means that it takes into account primarily very small tolerance distances among the data in a sample. By this token the test is able to uncover only a specific class of non-iid patterns within data.

The second interval is motivated by the results of Monte Carlo studies performed by Hsieh and LeBaron (1988), who found that the power and size of the BDS test is maximized when proximity parameter ε is chosen between 0.5 and 1.5 of the standard deviation of the sample. For this reason we have chosen to calculate values of correlation integral for (0.50σ, 1.50σ) range of proximity parameters. Table 1 illustrates that, indeed, numerous empirical studies used isolated values of proximity parameter within such a range of values.

The third interval represents the broadest range of sensible proximity parameter values that are used in the empirical literature. Since we concur with Kanzler (1999), who shows that the asymptotic normality of the BDS test depends on the correct choice of proximity parameter, we employ the broad interval of proximity parameters within (0.25σ, 2.00σ) in order to avoid omitting its possible correct values when computing the statistic of the Kočenda test. The ample use of proximity parameter values from within the third range is again documented in Table 1.

Further, based on the performance of the test for the three ranges indicated above, we intend to perform a Monte Carlo study to analyze the power of the test when previously detected "best" range is expanded or contracted. The ε -range over which the power of the test is ultimately maximized should be considered as the optimal one.

3.2 Critical values

In order to derive statistical properties of the Kočenda test when different proximity parameter ranges are used, a Monte Carlo study of the distribution of the statistic under the

range without any theoretical or empirical argument given to support the choice.

null hypothesis is performed.[10] Our sample consists of drawing 20000 time series from a standard normal distribution of length 500, 1000, and 2500 observations in each series.[11]

The data were generated using an inversive congruential generator. The first 1000 observations were discarded to avoid dependency on the initial condition. Finally, the generated data were randomly shuffled to reduce any hidden non-random dependencies in the data. The practical advantage of an inversive congruential generator (ICG) against a linear congruential generator (LCG) is that ICG guarantees the absence of a lattice structure. We have opted for the ICG for its superiority, despite the fact that it is significantly slower than LCG. Both generators are easy to implement and there is abundant literature available with the portable code, parameters and test results.[12] For a concise survey of the performance of inversive random number generators in theoretical and empirical tests, as well as tables of parameters to implement inversive generators see Hellekalek (1995). For a survey of the latest concepts and results of random number generation, we recommend starting with L'Ecuyer (2004).

Following Kočenda (2001), the generated iid samples were exposed to the computational procedure of the correlation integral allowing for nine embedding dimensions m (2-10) and 41 tolerance distances ε ranging over the three different intervals introduced in Section 3.1, in proportionally equal increments. Then, slope coefficient estimates of β_m were calculated according to equation (2.3a) along with the use of the cut-off point described in Section 2.

Finally, quantiles for the slope coefficient estimates β_m at different dimensional levels were tabulated. Table 2 presents the quantiles to allow a hypothesis testing at levels of 1%, 2%, 5%, and 10% for time series of 500, 1000, and 2500 observations for the ε-range (0.25σ, 1.00σ). Similarly, Tables 3 and 4 present the quantiles for ε-range (0.50σ, 1.50σ) and (0.25σ, 2.00σ), respectively. Let L_α and U_α be lower and upper bounds of the (100 - α) percentage confidence interval. If $[(x < L_\alpha) \vee (x > U_\alpha)]$, then the null hypothesis

[10] Monte Carlo simulations are used instead of distribution theory because the test is non-parametric.
[11] 5000 replications are used for sensitivity analysis among selected sub-ranges (see Section 4.3).
[12] The described data-generating strategy was chosen for two reasons. First, an ICG effectively eliminates repetitiveness in the data caused by the limitations of computer hardware. Secondly, other methods such as hypothetically obtaining white noise residuals by estimating a generating process (i.e. AR, ARCH, GARCH, etc.) may possess some unaccounted for structural form which would bias the critical values in a Monte Carlo simulation. The issues of how the asymptotic distribution of the test statistics might be affected by the estimation process is discussed by de Lima (1998).

of iid can be rejected at the α percent confidence level.

4. PERFORMANCE, POWER OF THE TEST AND OPTIMAL RANGE

4.1 Performance in Competition

In order to compare performance of the Kočenda test with other tests for nonlinearity and chaos we have exploited the study of Barnett et al. (1997) who performed a well-known single-blind controlled competition to compare the power of five highly regarded tests for nonlinearity or chaos against various alternatives.[13] This approach allows us to distinguish which of the three ε-ranges maximizes the power of the test as well. This way, selection of the appropriate range loses its mystery.

In order to facilitate accessibility of our results we briefly outline the above single-blind competition. Despite the fact that the study of Barnett et al. (1997) is widely known, we urge consulting the original source for exhaustive details. The data used in this competition were simulated by five different generating specifications at two sample sizes. The "small" sample size contains 380 and "large" sample size 2000 observations. The samples were generated by the following five models:

1. Fully deterministic, chaotic Feigenbaum recursion (FEIG) in the form of the logistic equation:

$$y_t = 3.57 y_{t-1}(1 - y_{t-1}), \tag{4.1}$$

where the initial condition was set at $y_0 = 0.7$;

2. A generalized autoregressive conditional heteroscedasticity model (GARCH) of the form:

$$y_t = h_t^{1/2} u_t, \tag{4.2}$$

where h_t is defined by $h_t = 1 + 0.1 y_{t-1}^2 + 0.8 h_{t-1}$, with $h_0 = 1$ and $y_0 = 0$;

3. A nonlinear moving average model (NLMA) of the form:

$$y_t = u_t + 0.8 u_{t-1} u_{t-2}; \tag{4.3}$$

4. An autoregressive conditional heteroscedasticity model (ARCH) of the form:

$$y_t = \left(1 + 0.5 y_{t-1}^2\right)^{1/2} u_t, \tag{4.4}$$

[13] We acknowledge that Brock, Hsieh, and LeBaron (1993) performed similar tests but these were done mainly on the BDS test and as such they are less suitable as a point of reference for our further purpose.

with the value of the initial observation set at $y_0 = 0$;

5. An autoregressive moving average model (ARMA) of the form:

$$y_t = 0.8 y_{t-1} + 0.15 y_{t-2} + u_t + 0.3 u_{t-1}, \qquad (4.5)$$

with $y_0 = 1$ and $y_1 = 0.7$.

The white noise disturbances, u_t, in the four stochastic models were sampled independently from a standard normal distribution and were generated using the fast acceptance-region algorithm of Kinderman and Ramage (1976), with the initial seed value set by the clock of the computer at the time the program was run. Of the five data generating models, specification (4.1) is chaotic and noise free, whereas the other specifications represent stochastic processes. [14]

The five tests that were used in the competition are the following: Hinich bispectral test in the frequency domain of flatness of the bispectrum, which is a test of the null hypothesis that the skewness function is flat, and hence that there is a lack of third order nonlinear dependence (for details see Hinich, 1982); BDS test for implicit evidence of nonlinearity. This is a test of the null hypothesis of iid (for details see Brock, Dechert, Scheinkman, and LeBaron, 1996); NEGM nonparametric test for positivity of the maximum Lyapunov exponent, which is a direct test for chaos (for details see Nychka, Ellner, Gallant, and McCaffrey, 1992); White test for nonlinearity, a test of the null hypothesis of the linearity in the mean (for details see White, 1989a, b; Lee, White, and Granger, 1993; Jungeilges, 1996); Kaplan test for nonlinearity, which is a test of the null hypothesis of linearity of the dynamics found in the data (for details see Kaplan, 1994).

Following the strategy of Barnett et al. (1997) we have used the same samples of the data that were used for the blind competition and run the Kočenda test on them; we have downloaded all ten data samples generated by models (4.1-4.5) from the Working Paper Archive maintained at Washington University.[15] The results are presented in columns 1-3 and 5-7 of Table 5.

For the most restrictive interval (0.25σ, 1.00σ) and small sizes of the data the test was unable to confidently reject the iid hypothesis in case of GARCH, NLMA, and ARCH processes. Poor performance of the test in the case of the ARCH process occurred at

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[14] For exhaustive details on models, data generating, as well as discussion on particular processes, see the original paper of Barnett et al. (1997).

[15] The web address of the data is http://econwpa.wustl.edu/eprints/data/papers/9510/9510001.abs

embedding dimensions of 6 and higher; this range of values is associated with a higher dispersion of critical values. For two additional intervals, (0.50σ, 1.50σ) and (0.25σ, 2.00σ), the test rejected the iid hypothesis correctly at the 1% level for all processes and both sample sizes. This result is very encouraging since rejection of the iid null could also be made at all embedding dimensions; thus, the procedure has not left a void in inference with respect to both subjectively selected parameters of the test. Note that both sample sizes of competition data (380 and 2000) are different than sizes for which we generated critical values (500, 1000, and 2500). For small (380) and large (2000) samples the closest critical values for sizes of 500 and 2500 observations were used, respectively.

To summarize: our results show that the test performance is very satisfactory since with a correctly selected ε-range it performs equally or better than the tests included in the competition performed by Barnett et al. (1997).[16] Further, the time needed to run the test with our new software is negligible.

4.2 Power Tests

In addition to the performed competition we decided to verify performance of the test with multiple data sets by proceeding with a series of power tests. Power tests allow for further assessment of performance as well as enable us to establish a basis for correct selection of the ε-range.

In order to show ability of the test to correctly distinguish between truly random and random-like data we have performed a series of power tests to judge the performance at a 5% significance level, thus fixing the probability of the „first-type“ error. When the test is applied to the random-like data, the relative number of non-rejections of the null

[16] The following summary of the competition results comes from from section *9.1 Overview* of Barnett et al. (1997). The Hinich bispectrum test was correct in three out of the five cases and failed in two of the cases with the small sample. With the large sample, the test was correct in three of the five cases, failed in one case, and was ambiguous in one case. The associated Gaussianity test, is a test of a necessary and not sufficient condition for Gaussianity and hence can reject but not accept. Judging the test on its rejections of Gaussianity, the small sample results produced only two rejections, and both were correct rejections. With the small sample, the test produced four rejections, and all four were valid rejections. With the small sample, the BDS test was correct in two cases out of five and ambiguous in the other three. With the large sample, the test was correct in all five cases. The NEGM test was correct in all five small sample cases and all five large sample cases. In the small sample cases, White's test was correct in four out of the five cases, and failed in the remaining case. In the large sample cases, White's test again was correct in four out of the five cases, and failed in one case. Kaplan's test was correct in all five cases both with the small samples and the large samples.

hypothesis (H_0: data are iid) at the given significance level corresponds to the probability that the test is subject to the „second-type" error—not rejecting the null hypothesis when it is not true. The smaller the probability of the „second-type" error (probability of the „first-type" error being fixed) is, the greater the power of the test.

To conduct the power tests we pursued the following strategy. For each of the five models described in specifications (4.1-4.5), and used by Barnett et al. (1997), we have generated 1000 samples of data.[17] The data were generated in three sizes of 500, 1000, and 2500 observations. We have used all samples to perform a battery of power tests; the results are reported in Table 6. For better accessibility we report cumulative results of the power tests as a percentage of H_0 rejections; the percentage when the test correctly rejects the null hypothesis since tested data are anything but iid.

Across all proximity ranges as well as sample sizes the test always accurately rejects the null hypothesis for FEIG and ARMA processes. For the remaining processes the power of the test uniformly improves with the sample size as one would expect.[18] GARCH, ARCH, and NLMA processes pose some challenge to the test at sample size of 500 observations; the power of the test dramatically improves for ARCH and NLMA processes when samples of 1000 and 2500 observations are used. An interesting picture emerges when the power of the test is compared among the three ranges of the proximity parameter. The power is lowest for interval (0.25σ, 1.0σ) but improves radically for intervals (0.5σ, 1.5σ) and (0.25σ, 2.0σ). For these two intervals the power of the test is extremely high, above 90% in most cases even for the small sample size of 500 observations; an exception is the power in the GARCH process. The power of the test is highest for both wide intervals and large sample size (2500 observations) in all five processes; it is near 100% of correct rejections for the GARCH process and precisely 100% for remaining processes.

Range (0.50σ, 1.50σ) has slightly better power over embedding dimensions $m = 2$-5 than the range (0.25σ, 2.00σ), which has better power over dimensions $m = 6$-9. Since testing at high levels of embedding dimension m (when m is higher than 5) is often questionable due to reasons articulated earlier, the range (0.50σ, 1.50σ) may be considered

[17] Because the Feigenbaum process is deterministic we have replicated 1000 times only the four other processes, to be precise. Since competition performed by Barnett et al. (1997), understandably, does not contain power tests of participating tests, we do not offer any comparison in this respect.

[18] This finding is in line with results reported by Hsieh and LeBaron (1991) who have found that type I error

as the best option out of the three ranges discussed so far. We now proceed with analysis of how the power of the test changes when the extent of the range is slightly modified.

4.3 Sensitivity Analysis and Optimal Range

4.3.1 Optimal range

In the previous section we have shown which of the chosen ranges performs better than the others. We now analyze whether the power of the test improves while the given range is expanded or contracted. Since the range (0.50σ, 1.50σ) is a subset of the range (0.25σ, 2.00σ) and the range (0.50σ, 1.50σ) came out as the best option for the lower embedding dimensions (core of the applied research), such an approach can be considered as a robustness check of the range.

The sensitivity analysis of the power of the test against nonlinear non-iid processes (as in Section 4.2) was pursued with the following strategy. First, we have expanded the range (0.50σ, 1.50σ) by small amounts (0.10σ) from the lower and upper bound separately. Expansion reached 0.30σ and 2.10σ at the lower and upper bound, respectively. Second, we have expanded the range (0.50σ, 1.50σ) in a similar fashion from the lower and upper bound at the same time. Third, we have contracted the range (0.50σ, 1.50σ) from one end separately and later from both ends at the same time as well. Finally, we have contracted the range (0.50σ, 1.50σ) from one end and expanded it from the opposite end. This effectively meant, shifting the range. The upper limit we tested reached 3.00σ, the lower limit was 0.25σ. The detailed results from this analysis are not presented due to their excessive amount (they are available upon request).

The results of the above exercise can be summarized and generalized in the following way: by contracting the range the test was losing its power while by expanding the range the power was increasing; such an increase in power was limited, though. Beyond the range (0.60σ, 1.90σ) the power ceased to increase or its increase was negligible, depending on a specific embedding dimension. Hence, we consider the range (0.60σ, 1.90σ) of the tolerance distance ε to be the optimal range for conducting the test. Table 7 contains the critical values tabulated for the ε-range (0.60σ, 1.90σ).

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is large with the BDS test when the sample size is small.

4.3.2 Sensitivity of critical values to a ε-range choice

Kočenda and Briatka (2004) compared the variation in critical values (at a given sample size and embedding dimension) for the three ε-ranges introduced in Section 3.1. The main result of the sensitivity analysis is that the critical values differ for the three ε-ranges and these differences are smallest between the critical values of the ranges [(0.50σ, 1.50σ) and (0.25σ, 2.00σ)] (see Kočenda and Briatka, 2004, for exhaustive details).

We now present the sensitivity of critical values to a ε-range choice in the sense that we compare the optimal range (0.60σ, 1.90σ) with the three ranges introduced in Section 3.1. Table 8 shows dispersions, or relative changes of critical values between ranges, that are calculated as respective quantile differences between critical values obtained for the optimal range and another specified range at a particular embedding dimension. Further, Figure 1 shows the same number as the relative change in absolute value. The relative changes are of various signs but in Figure 1 they are pictured in absolute values for better accessibility.

In general, the smallest differences of critical values are between pairs of ranges [(0.60σ, 1.90σ) and (0.50σ, 1.50σ)] in the low embedding dimensions m = 2, 3, 4, and 5; and between ranges [(0.60σ, 1.90σ) and (0.25σ, 2.00σ)] in the high embedding dimension m = 6, 7, 8, 9, and 10. The relative differences for these ranges and dimensions are lower than 8% throughout the whole sample. In contrast, the highest differences are between ranges [(0.60σ, 1.90σ) and (0.25σ, 1.00σ)]; in the most pronounced case this difference is almost 41%.

The variation of critical values is not only associated with the range of proximity parameters but also with the value of the embedding dimension used: the differences of critical values tend to increase with embedding dimension m. The main reason is that as embedding dimension m increases, fewer and fewer non-overlapping m-histories become available to compute the sample correlation integral. In a small sample, there are few data available and the deviation of critical values is thus greater for a small data size.[19]

A further implication is that for samples of moderate size only a low-dimensional chaos is characterized. This is due to the fact that the estimates of $C_{1,T}(\varepsilon)$ show a far larger

[19] This is in line with the findings of Brock, Hsieh, and LeBaron (1993) and Kanzler (1999) with respect to the BDS test: as the embedding dimension m increases, the BDS distribution moves away from its asymptotic distribution, the standard normal. The lower the dimension, the better the small-sample properties, whatever the sample size and size of the ε.

deviation around their asymptotic values when the samples are small than when they are large (Kanzler 1999). However, as we stated earlier, the correct choice of ε is unknown. Moreover, small ε causes the expected number of close histories to be small and thus renders estimation of $C_{m,T}(\varepsilon)$ less reliable. This means that the choice of ε has a greater impact on the reliability of $C_{m,T}(\varepsilon)$ than on $C_{1,T}(\varepsilon)$.

4.3.3 Competition and power

Additionally, we have again used the samples of the data that were used for the blind competition of Barnett et al. (1997) and run the Kočenda test with the new optimal range on them. The results are presented in columns 4 and 8 of Table 5. The test rejected the null hypothesis of iid for the whole set of data and rejected the iid hypothesis with higher statistical significance than when performed with the three earlier ranges. The results also show superior performance of the test with respect to other tests for nonlinearity (as in Section 4.1). Further, similarly to the power tests that compare performance under different ranges (introduced in Section 4.2), we report the results of the power tests that were performed with the optimal range (0.60σ, 1.90σ); results are given in Table 9 and show superior performance of the test over other ranges compared in Section 4.2.

4.4 Further Power Tests on Chaotic Data with Noise

For the sake of consistency the previous power tests were performed on the set of processes used in Barnett at al. (1997). We performed further power tests on other chaotic processes with additive noise. In order to keep the extent of work manageable we have chosen two standard non-univariate chaotic systems to generate the data; both were additively contaminated with noise.

First, we have chosen the Hénon (1976) map, which is a bivariate chaotic system described by a pair of difference equations; for our purpose we used a collapsed version of the original system given by specification as in Lai and Chen (2003):

$$y_{t+1} = 1 + 0.3y_{t-1} - 1.4y_t^2 \qquad (4.6)$$

which was further contaminated by noise. The new noisy chaotic series x_t was constructed as $x_t = y_t + u_t$ where y_t satisfies (4.6) and u_t is independently and normally distributed with mean 0 and variance 0.04.

The test correctly recorded a 100% rejection rate of the iid hypothesis for both chaotic and noisy chaotic series for all four ε-ranges used (the results are not presented here for the sake of space limitations but are available upon request).

Further, we opted for the Lorenz (1963) map, which is a trivariate chaotic system described by a system of differential equations. However, for our purpose we used a more intricate specification of the system of 10 differential equations suggested by Lorenz (1996) and Emanuel and Lorenz (1998). In general, it is a system of N differential equations with N variables specified as:

$$\dot{x}_i = -(x_{i-2} + x_{i+1})x_{i-1} - x_i + f \text{, where } i = 1, 2, ..., N. \quad (4.7)$$

To make these equations meaningful, it was set $x_{-1} = x_{N-1}$, $x_0 = x_N$ and $x_{N+1} = x_1$. The variables of the system are scaled so that the coefficients of the quadratic and linear terms are unity. For the purpose of our simulation we consider $N = 10$. Like the previous case of the Hénon map we contaminated data with noise by adding error process u_t, where u_t is independently and normally distributed with mean 0 and variance 0.04 as in Lai and Chen (2003).

The test correctly recorded a 100% rejection rate of the iid hypothesis for both chaotic and noisy chaotic series for all four ε-ranges used (the detailed results are available upon request).

4.5 Range Selection Recommendation

The results of the power tests combined with the findings on sensitivity of the critical values to the ε-range choice enable us to formulate the following recommendations.

Unless assumptions of a research project dictate otherwise, the interval (0.60σ, 1.90σ) is considered as optimal and should be used as a template option. If no preferences associated with research motivations are set, we suggest avoiding the use of the (0.25σ, 1.00σ) range since the power of the test is lower than for the other intervals (negatively biased estimator). Ranges (0.50σ, 1.50σ) and (0.25σ, 2.00σ) should be used for tests carried out under specific research requirements related to the ε-range span.

4.6 When the Bootstrap is Needed

The critical values in Sections 3.2 and 4.3 were tabulated under the assumption of a standard normal distribution. This is a sensible assumption when real data are meant to be scrutinized by the test. The two kinds of most widely used data exploited in the literature on nonlinear dynamics are exchange rates and stock prices (see Hsieh, 1989 and 1991 for comprehensive assessments); further typical data are returns on bonds and treasury as well as inflation rates (see Hiemstra and Kramer, 1997). Hence, for most of the real economic data, the use of tabulated critical values is wholly appropriate. However, this does not need to be the case if data exhibit excessive departure from standard normal. Belaire-Franch (2003) has shown that if there is a large excess kurtosis in the data (α=1.2), the iid hypothesis would be erroneously rejected by the test very frequently. He argues that the bootstrapping method (the random shuffle) should be used to generate particular critical values for each series analyzed to avoid asymptotically biased critical values. Specifically, the given sample of size T is taken as the population and samples of size T are drawn from the "population" with replacement. The samples are used to obtain an estimate, called the bootstrap distribution of the true population distribution. By replacing the true population distribution by the bootstrap distribution we may obtain an estimate of the distribution of an estimator or test statistic of interest.

The departure from normality due to a large kurtosis (as in Belaire-Franch, 2003) or contaminated normal distribution with a very large contaminating variance (e. g. K^2=100 as in Ronchetti and Trojani, 2003)[20], certainly justifies using the bootstrap to generate custom-made critical values in the case of artificial, computer-generated time series. Such radical departures from normality are not typical for real economic data or the residuals coming from nonlinear models that use such economic data. For that reason, the significance of Kočenda's statistic should be evaluated on quantiles generated from the standard normal distribution only when the departure of tested time series from the normal distribution is very small. We measure the extent of such departure by using a normality test, e.g. the tests for kurtosis and skewness of the distribution, along with the Bera-Jarque

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[20] Ronchetti and Trojani (2003) used data generated from contaminated normal distribution CN(ε, K^2) given by distribution function $F(x) = (1-\varepsilon)\ \Phi(x) + \varepsilon\ \Phi(x/K),\ x \in R$, where $\Phi(x)$ is a cumulative distribution function of a standard normal random variable.

normality test.[21] In other cases we suggest using the bootstrap method as discussed in Belaire-Franch (2003).

5. PERFORMANCE OF THE TEST WITH REAL DATA

As a complementary analysis to the performance of the test with chaotic artificial data we have also performed analysis with real data. For the comparative purpose as well as that of replication accuracy we used the data on exchange rates employed in the studies by Kugler and Lenz (1990, 1993) and Brock, Hsieh and LeBaron (1993); for the sake of consistency we use the same notation as in those studies. These data samples were used earlier by Kočenda (2001) and Belaire-Franch (2003), which allows some comparison.

5.1 Analysis of ARCH corrected weekly exchange rates

Kugler and Lenz (1990) analyzed nonlinear dependence of weekly exchange rate changes for four currencies against the US dollar from 1979 to 1989 (575 observations, the rate of change of the log exchange rate $x_t = \Delta \log S_t$). The data were corrected to account for the present ARCH process by transformation into the ARCH corrected rate of changes in the form

$$\Delta \log S_t^h = \Delta \log S_t \Big/ \left(\hat{\alpha}_0 + \sum_{\tau=1}^{6} \hat{\alpha}_\tau \Delta \log S_{t-\tau}^2 \right)^{0.5} \qquad (5.1)$$

where α-coefficients were obtained by OLS regression of $(\Delta \log S_t)^2$ on constant and six lagged variables.[22]

We have replicated the original study of Kugler and Lenz (1990) with the same results as Kočenda (2001). The results are presented in Table 10 and as in Kočenda (2001) or Belaire-Franch (2003) the null hypothesis is rejected for the French Franc (FRF), Japanese Yen (JPY) and the Deutsche Mark (DEM). The Swiss Franc (CHF) is the only currency where the null of iid cannot be rejected.

[21] This information is conveniently provided when employing our new software.

[22] Kugler and Lenz (1990) found that the described correction successfully removed nonlinearity from the Swiss Franc and Deutsche Mark. However, the BDS test did not allow rejection of the null hypothesis for the French Franc (specifically at levels of N = 4 and 5) and Japanese Yen (specifically at levels of N = 3, 4, and 5).

5.2 Analysis of daily exchange rates

Brock, Hsieh, and LeBaron (1993) analyzed the daily closing bids for five major currencies in U.S. Dollars: Swiss Franc (CHF), Canadian Dollar (CAD), Deutsche Mark (DEM), British Pound (GBP), and Japanese Yen (JPY)[23] during the period from January 2, 1974 to December 30, 1983 (2,510 observations). The specification of the model resulted in the mean equation

$$r_t = \beta_0 + \sum_{i=1}^{j} \beta_i r_{i-1} + \beta_M D_{M,t} + \beta_T D_{T,t} + \beta_W D_{W,t} + \beta_R D_{R,t} + \beta_H D_H + u_t \qquad (5.2a)$$

where, $u_t | \Omega_{t-1} \sim D(0, h_t)$, and variance equation

$$h_t = \phi_0 + \psi u_{t-1}^2 + \phi h_{t-1} + \phi_M D_{M,t} + \phi_T D_{T,t} + \phi_W D_{W,t} + \phi_R D_{R,t} + \phi_H D_H \qquad (5.2b)$$

where r_t is the rate of change of the nominal exchange rate at time *t*, $D_{M,t}$, $D_{T,t}$, $D_{W,t}$, and $D_{R,t}$, are dummy variables for days in a week (Monday, Tuesday, Wednesday, and Thursday) and D_H is the number of holidays between two successive trading days excluding week-ends. The order of the AR process was found to be *j* = *6*, *5*, *6*, and *0* respectively for CHF, CAD, DEM, and GBP.

After estimation, the Kočenda test is run on standardized residuals $z_t = u_t / h_t^{\frac{1}{2}}$. The results are presented in Table 11 and the null hypothesis is rejected for all four time series.[24]

5.3 Analysis of weekly exchange rates

Kugler and Lenz (1993) analyzed the non-linear dependence of exchange rate changes for ten currencies against the US dollar. The sample period is from 1979 to 1989 (575 observations, the rate of change of the log exchange rate $x_t = \Delta \log S_t$). In order to check

[23] Japanese Yen was dropped from the replication because of data inconsistency.

[24] In Brock, Hsieh, and LeBaron (1993) the BDS test finds no evidence of nonlinearity in standardized residuals of CHF, some nonlinearity (at dimensions 8, 9, and 10) for the DEM, and strong nonlinearity for CAD and GBP; Belaire-Franch (2003) concurs with this result. Kočenda (2001) found that DEM and GBP show the presence of nonlinearity at the 1% significance level no matter what embedding dimension is considered. CAD and CHF show some presence of nonlinearity at various significance levels depending on embedding dimension *m*.

whether the detected dependence can be attributed solely to an ARCH process, the authors estimated the following GARCH-M model

$$\Delta \log S_t = \beta_0 + \sum_{\tau=1}^{3} \beta_\tau \Delta \log S_{t-\tau} + \beta_4 \sqrt{h_t} + \eta_t \tag{5.3}$$

$$h_t = \alpha_0 + \alpha_1 \eta_{t-1}^2 + \alpha_2 h_{t-1} \qquad \eta_t = \varepsilon_t \sqrt{h_t} \,.$$

After estimation the fitted residuals $\hat{\varepsilon}_t = \eta_t / \sqrt{h_t}$ are subjected to the test.[25] The results of the test with the new interval (0.60σ, 1.90σ) are presented in Table 12 and provide even stronger results than in Kočenda (2001). The null hypothesis of iid is rejected in all cases but one: only the Swiss Franc is still shown to be without any nonlinear dependency.

6. CONCLUSION

In this paper we extend the test of Kočenda (2001) and deliver two improvements to the operational ability of the procedure.

First, we improve the choice of the range of the proximity parameter ε over which the correlation integral is calculated, tabulate new sets of critical values for various lengths of data, and provide a sensitivity check to the robustness of critical values with respect to the choice of range of proximity parameters. We perform a series of power tests and suggest the range that maximizes the power of the test. Unless assumptions of a particular research project dictate otherwise, the interval (0.60σ, 1.90σ) for the ε-range choice should be used as a template option.

Second, we compare the test with existing results of the single-blind controlled competition of Barnett et al. (1997) and additionally perform power tests on various chaotic (nonlinear) and noisy chaotic data. The results strongly attest to our robust procedure. The improvement allows for new economic insights especially when the procedure is used on financial data as a specification test. Comparison with real exchange rate data is provided as well.

As a final contribution, we introduce a new compact program to run the test as well as

[25] In Kugler and Lenz (1993) results of the BDS-test revealed no indication of dependence in the fitted residuals of any currency. Kočenda (2001) confirmed the findings of independence for 5 of the 10 currencies (CAD, BEF, FRF, NLG, and CHF) and detected nonlinear dependencies in the fitted residuals for the rest of the supposedly independent currencies (AUD, DEM, ITL, ESP, and JPY). Belaire-Franch (2003) did not analyze this data.

simulations (see Appendix). The software is very fast, user-friendly, and may be downloaded from http://home.cerge-ei.cz/kocenda/software/ as freeware, subject to appropriate citation.

ACKNOWLEDGMENTS

We would like to thank William Barnett, Tim Bollerslev, Jan Hanousek, Blake LeBaron, Elzevio Ronchetti, Jan Ámos Víšek, Petr Zemčík, and the Editor (Esfandiar Maasoumi) for helpful comments. We are grateful to an anonymous referee whose comments and suggestions helped to considerably improve the paper. We also benefited from several presentations. We thank Marian Baranec, Ivo Burger, Eva Čermáková, Petr Sklenář and the CERGE-EI Computer Department for their assistance to perform simulations in this paper. The usual disclaimer applies.

SOFTWARE APPENDIX

In order to facilitate running the test we provide fast and user-friendly software to calculate values of the test statistics. In line with the previous exposition, a researcher is offered an optimal range of epsilons (0.60σ; 1.90σ) as a template option (further, as a specific option, that is subject to requirements of a research project, a user has three other choices for range of epsilons (0.25σ; 1.00σ), (0.50σ; 1.50σ) or (0.25σ; 2.00σ)).

The test statistic is computed at nine embedding dimensions $m = 2, \ldots, 10$. The program automatically suggests the optimal set of critical values based on the sample size, compares the computed values with built-in critical values (those reported in Tables 7 or alternatively in Tables 2-4) and notes statistical significance as well as “reject/no reject” result.

There is also a built-in data diagnostic panel, which provides description of the data: number of observations, mean, standard deviation, spread and the standard deviation divided by the spread. As a test for normality of the input data values of kurtosis, skewness, and Bera-Jarque normality tests are provided as well. For an even more detailed view of the series tested, one can use the histogram button: a new pop-up window opens and a histogram is plotted.

An advanced menu in the program allows one to compute a test statistic for arbitrary values of ε-range. Please note that in this case the built-in critical values should not be used and it is suggested to proceed with the bootstrapping method in order to generate an appropriate set of critical values. For more experienced users and those interested in doing simulations, a fast engine was compiled separately. It can be used to perform Monte Carlo studies or can be built in as a part of some other program.

The program is user-friendly with a self-explanatory design. The program requires a Windows 98/2000/NT/XP environment. It is posted on our webpages as freeware, subject to appropriate citation. More information can be found in a "readme" file that is part of the program. No warranty of any kind is expressed or implied. No liability is assumed for the use of the software.

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Table 1
Summary of ε-ranges used in selected studies

Year	The author of the study	Used values of proximity parameter ε (ε as a fraction of standard deviation)
1989	Hsieh	0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 2.00
1990	Kugler and Lenz	0.50, 0.75, 1.00, 1.50
1991	Hsieh	0.25, 0.50, 1.00, 1.50, 2.00
1992	Rothman	0.50, 1.00, 1.25, 1.50, 2.00
1993	Kugler and Lenz	1.00
1993	Hsieh	0.50, 1.00, 1.50, 2.00
1993	Brock, Hsieh, and LeBaron	0.50, 0.75, 1.00, 1.25, 1.50
1996	Brock, Dechert, Scheinkman, and LeBaron	0.50
1996	Cecen and Erkal	0.50
1996	de Lima	1.00, 1.25
1996	Chappell, Padmore, and Ellis	0.40, 0.625, 1.00, 1.60
1996	Kočenda	0.50, 1.00
1997	Krämer and Runde	1.00
1997	Serletis and Gogas	0.50, 1.00, 1.50, 2.00
1998	Johnson and McClelland	1.00
1998	Chwee	0.50, 1.00, 1.50, 2.00
1998	de Lima	0.50, 0.75, 1.00, 1.25, 1.50
1999	Mahajan and Wagner	0.50, 0.75, 1.00, 1.25, 1.50
1999	Opong, Mulholland, Fox, and Farahmand	0.50,1.00, 1.50, 2.00
1999	Brooks	1.00, 1.50
1999	Brooks and Heravi	0.50, 1.00, 1.50, 2.00
2000	Brooks and Henry	1.00, 1.50
2000	Andreou, Pavlides, and Karytinos	0.50, 1.00, 1.50, 2.00
2000	Aguirre and Aguirre	0.65, 0.70, 0.75, 0.80, 0.85, 0.90
2001	McKenzie	0.50, 1.00, 1.50, 2.00
2001	Bodman	1.00
2002	Díaz, Grau-Carles, and Mangas	1.50
2002	Chen and Kuan	0.75, 1.00
2003	Diks	several values between 0.50 and 1.50
2003	Panagiotidis	0.50, 1.00, 2.00
2004	Muckley	0.50, 1.00

Table 2

Quantiles of the slope coefficients β_m for ε-range (0.25σ – 1.00σ)

A. Quantiles of the slope coefficients β_m for a sample size of 500 observations

Quantile	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
0.5%	1.850	2.736	3.547	4.252	4.897	5.418	5.787	5.934	5.531
1.0%	1.857	2.749	3.570	4.293	4.952	5.506	5.909	6.092	5.829
2.5%	1.865	2.767	3.610	4.353	5.036	5.629	6.092	6.354	6.185
5.0%	1.872	2.782	3.639	4.407	5.109	5.742	6.241	6.572	6.540
95.0%	1.929	2.923	3.955	4.909	5.885	6.895	8.006	9.322	11.209
97.5%	1.933	2.936	3.987	4.957	5.965	7.017	8.202	9.660	11.997
99.0%	1.938	2.952	4.019	5.010	6.050	7.167	8.435	10.087	13.026
99.5%	1.941	2.963	4.041	5.046	6.107	7.279	8.611	10.425	13.747

Note: "*m*" denotes an embedding dimension. Based on 20,000 replications.

B. Quantiles of the slope coefficients β_m for a sample size of 1000 observations

Quantile	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
0.5%	1.875	2.793	3.669	4.464	5.192	5.844	6.420	6.870	7.208
1.0%	1.878	2.799	3.682	4.486	5.225	5.898	6.501	6.996	7.358
2.5%	1.882	2.808	3.701	4.521	5.279	5.984	6.620	7.149	7.579
5.0%	1.886	2.816	3.718	4.557	5.329	6.051	6.715	7.293	7.765
95.0%	1.916	2.888	3.893	4.890	5.826	6.770	7.747	8.777	9.905
97.5%	1.918	2.895	3.911	4.922	5.870	6.834	7.849	8.917	10.148
99.0%	1.921	2.902	3.931	4.960	5.927	6.908	7.966	9.096	10.458
99.5%	1.922	2.906	3.945	4.985	5.970	6.953	8.041	9.242	10.695

Note: "*m*" denotes an embedding dimension. Based on 20,000 replications.

C. Quantiles of the slope coefficients β_m for a sample size of 2500 observations

Quantile	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
0.5%	1.889	2.826	3.745	4.625	5.418	6.183	6.897	7.544	8.121
1.0%	1.890	2.828	3.751	4.638	5.443	6.221	6.940	7.618	8.212
2.5%	1.892	2.832	3.759	4.656	5.483	6.269	7.006	7.698	8.331
5.0%	1.894	2.836	3.767	4.671	5.513	6.309	7.067	7.775	8.436
95.0%	1.909	2.868	3.840	4.842	5.819	6.737	7.655	8.590	9.539
97.5%	1.910	2.871	3.846	4.856	5.847	6.779	7.711	8.666	9.657
99.0%	1.911	2.874	3.855	4.878	5.881	6.826	7.782	8.755	9.786
99.5%	1.912	2.876	3.859	4.892	5.898	6.857	7.825	8.818	9.880

Note: "*m*" denotes an embedding dimension. Based on 20,000 replications.

Table 3

Quantiles of the slope coefficients β_m for ε-range (0.50σ – 1.50σ)

A. Quantiles of the slope coefficients β_m for a sample size of 500 observations

Quantile	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
0.5%	1.678	2.503	3.310	4.093	4.831	5.447	6.011	6.489	6.959
1.0%	1.686	2.517	3.332	4.124	4.876	5.506	6.072	6.592	7.086
2.5%	1.696	2.534	3.360	4.163	4.931	5.581	6.164	6.717	7.228
5.0%	1.704	2.548	3.382	4.195	4.979	5.642	6.245	6.822	7.361
95.0%	1.766	2.659	3.566	4.498	5.430	6.220	7.003	7.788	8.577
97.5%	1.770	2.667	3.581	4.525	5.467	6.274	7.075	7.878	8.699
99.0%	1.774	2.676	3.598	4.560	5.511	6.340	7.153	7.984	8.835
99.5%	1.777	2.682	3.609	4.583	5.543	6.387	7.219	8.061	8.936

Note: "*m*" denotes an embedding dimension. Based on 20,000 replications.

B. Quantiles of the slope coefficients β_m for a sample size of 1000 observations

Quantile	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
0.5%	1.698	2.540	3.372	4.189	4.987	5.736	6.391	6.992	7.543
1.0%	1.703	2.548	3.386	4.209	5.011	5.770	6.438	7.041	7.610
2.5%	1.709	2.559	3.402	4.233	5.047	5.821	6.501	7.120	7.709
5.0%	1.715	2.567	3.415	4.253	5.076	5.868	6.555	7.186	7.792
95.0%	1.756	2.639	3.528	4.428	5.351	6.278	7.063	7.826	8.594
97.5%	1.759	2.644	3.537	4.443	5.378	6.314	7.107	7.893	8.672
99.0%	1.762	2.649	3.546	4.460	5.406	6.355	7.156	7.964	8.758
99.5%	1.764	2.653	3.552	4.471	5.426	6.378	7.193	8.003	8.813

Note: "*m*" denotes an embedding dimension. Based on 20,000 replications.

C. Quantiles of the slope coefficients β_m for a sample size of 2500 observations

Quantile	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
0.5%	1.715	2.568	3.418	4.262	5.095	5.914	6.702	7.416	8.045
1.0%	1.717	2.572	3.423	4.270	5.108	5.935	6.728	7.448	8.091
2.5%	1.721	2.578	3.433	4.283	5.127	5.958	6.763	7.498	8.156
5.0%	1.724	2.583	3.440	4.294	5.142	5.978	6.793	7.537	8.203
95.0%	1.748	2.625	3.504	4.387	5.277	6.181	7.117	7.938	8.696
97.5%	1.750	2.628	3.509	4.394	5.290	6.200	7.151	7.974	8.742
99.0%	1.752	2.632	3.515	4.403	5.303	6.222	7.188	8.014	8.797
99.5%	1.754	2.634	3.518	4.409	5.313	6.237	7.221	8.048	8.837

Note: "*m*" denotes an embedding dimension. Based on 20,000 replications.

Table 4

Quantiles of the slope coefficients β_m for ε-range (0.25σ – 2.00σ)

A. Quantiles of the slope coefficients β_m for a sample size of 500 observations

Quantile	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
0.5%	1.720	2.561	3.369	4.024	4.624	5.158	5.677	6.155	6.600
1.0%	1.726	2.572	3.387	4.053	4.655	5.208	5.728	6.209	6.679
2.5%	1.734	2.588	3.412	4.086	4.701	5.271	5.800	6.302	6.781
5.0%	1.741	2.599	3.433	4.110	4.734	5.319	5.860	6.378	6.864
95.0%	1.794	2.706	3.612	4.360	5.066	5.742	6.398	7.029	7.649
97.5%	1.798	2.715	3.629	4.381	5.095	5.782	6.444	7.080	7.714
99.0%	1.802	2.725	3.646	4.404	5.127	5.823	6.493	7.140	7.783
99.5%	1.804	2.733	3.660	4.420	5.152	5.856	6.523	7.185	7.834

Note: "*m*" denotes an embedding dimension. Based on 20,000 replications.

B. Quantiles of the slope coefficients β_m for a sample size of 1000 observations

Quantile	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
0.5%	1.737	2.597	3.437	4.217	4.876	5.478	6.056	6.596	7.099
1.0%	1.740	2.603	3.449	4.233	4.897	5.502	6.088	6.634	7.150
2.5%	1.746	2.612	3.465	4.256	4.924	5.546	6.135	6.695	7.219
5.0%	1.750	2.619	3.477	4.274	4.948	5.580	6.176	6.743	7.278
95.0%	1.785	2.684	3.598	4.448	5.179	5.877	6.547	7.196	7.827
97.5%	1.787	2.689	3.609	4.465	5.200	5.903	6.578	7.238	7.875
99.0%	1.790	2.695	3.623	4.483	5.222	5.932	6.612	7.283	7.926
99.5%	1.792	2.698	3.630	4.494	5.238	5.952	6.635	7.313	7.961

Note: "*m*" denotes an embedding dimension. Based on 20,000 replications.

C. Quantiles of the slope coefficients β_m for a sample size of 2500 observations

Quantile	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
0.5%	1.750	2.622	3.487	4.336	5.110	5.790	6.425	7.025	7.594
1.0%	1.752	2.625	3.492	4.344	5.125	5.804	6.447	7.046	7.627
2.5%	1.755	2.630	3.500	4.356	5.144	5.826	6.475	7.085	7.668
5.0%	1.758	2.634	3.506	4.366	5.159	5.844	6.497	7.113	7.706
95.0%	1.778	2.670	3.567	4.477	5.309	6.034	6.736	7.399	8.055
97.5%	1.780	2.673	3.572	4.487	5.322	6.052	6.756	7.423	8.090
99.0%	1.781	2.676	3.577	4.499	5.339	6.073	6.779	7.454	8.127
99.5%	1.783	2.678	3.581	4.507	5.351	6.090	6.797	7.479	8.153

Note: "*m*" denotes an embedding dimension. Based on 20,000 replications.

Table 5
Computed Kočenda test statistics for five processes taken from Barnett et al. (1997)

		The sample size T = 380				The sample size T = 2000			
		1	2	3	4	5	6	7	8
		(0.25σ-1.00σ)	(0.50σ-1.50σ)	(0.25σ-2.00σ)	(0.60σ-1.90σ)	(0.25σ-1.00σ)	(0.50σ-1.50σ)	(0.25σ-2.00σ)	(0.60σ-1.90σ)
FEIG	β_2	0.627 [a]	0.652 [a]	0.602 [a]	0.547 [a]	0.608 [a]	0.647 [a]	0.587 [a]	0.536 [a]
	β_3	0.629 [a]	0.736 [a]	0.590 [a]	0.539 [a]	0.606 [a]	0.733 [a]	0.571 [a]	0.526 [a]
	β_4	0.634 [a]	0.836 [a]	0.634 [a]	0.652 [a]	0.604 [a]	0.832 [a]	0.611 [a]	0.642 [a]
	β_5	0.609 [a]	0.873 [a]	0.653 [a]	0.737 [a]	0.573 [a]	0.863 [a]	0.627 [a]	0.725 [a]
	β_6	0.579 [a]	0.913 [a]	0.673 [a]	0.830 [a]	0.538 [a]	0.897 [a]	0.643 [a]	0.816 [a]
	β_7	0.584 [a]	0.913 [a]	0.691 [a]	0.852 [a]	0.544 [a]	0.894 [a]	0.660 [a]	0.835 [a]
	β_8	0.589 [a]	0.912 [a]	0.710 [a]	0.876 [a]	0.549 [a]	0.890 [a]	0.679 [a]	0.857 [a]
	β_9	0.614 [a]	0.914 [a]	0.728 [a]	0.883 [a]	0.577 [a]	0.889 [a]	0.696 [a]	0.859 [a]
	β_{10}	0.641 [a]	0.915 [a]	0.747 [a]	0.889 [a]	0.605 [a]	0.887 [a]	0.714 [a]	0.862 [a]
GARCH	β_2	1.840 [a]	1.685 [b]	1.709 [a]	1.556 [c]	1.894	1.712 [a]	1.749 [a]	1.578 [a]
	β_3	2.793	2.504 [b]	2.565 [b]	2.291 [a]	2.838	2.541 [a]	2.607 [a]	2.334 [a]
	β_4	3.673	3.318 [b]	3.359 [a]	3.017 [a]	3.754 [c]	3.362 [a]	3.446 [a]	3.082 [a]
	β_5	4.731	4.174 [d]	4.065 [c]	3.737 [a]	4.621 [a]	4.174 [a]	4.254 [a]	3.813 [a]
	β_6	5.650	5.027	4.596 [a]	4.488 [a]	5.451 [c]	4.989 [a]	5.018 [a]	4.545 [a]
	β_7	6.091	5.449 [b]	5.020 [a]	5.151 [a]	6.318	5.741 [a]	5.652 [a]	5.259 [a]
	β_8	6.505	5.954 [a]	5.520 [a]	5.622 [a]	6.983 [c]	6.511 [a]	6.193 [a]	5.973 [a]
	β_9	7.179	6.488 [a]	5.992 [a]	6.106 [a]	7.764 [d]	7.323 [a]	6.764 [a]	6.681 [a]
	β_{10}	8.835	7.110 [c]	6.416 [a]	6.596 [a]	8.308 [c]	7.887 [a]	7.288 [a]	7.351 [a]
NLMA	β_2	1.872	1.640 [a]	1.705 [a]	1.504 [a]	1.877 [a]	1.679 [a]	1.720 [a]	1.533 [a]
	β_3	2.795	2.418 [a]	2.529 [a]	2.209 [a]	2.794 [a]	2.470 [a]	2.542 [a]	2.244 [a]
	β_4	3.664	3.214 [a]	3.296 [a]	2.923 [a]	3.715 [a]	3.247 [a]	3.361 [a]	2.941 [a]
	β_5	4.345 [c]	3.941 [a]	3.862 [a]	3.596 [a]	4.571 [a]	3.999 [a]	4.137 [a]	3.623 [a]
	β_6	4.975 [c]	4.699 [a]	4.407 [a]	4.298 [a]	5.442 [b]	4.758 [a]	4.904 [a]	4.315 [a]
	β_7	5.791	5.409 [a]	4.993 [a]	5.038 [a]	6.126 [a]	5.498 [a]	5.417 [a]	5.007 [a]
	β_8	6.918	6.129 [c]	5.579 [a]	5.670 [a]	6.742 [a]	6.229 [a]	5.966 [a]	5.698 [a]
	β_9	7.974	6.723 [d]	5.994 [a]	6.138 [a]	7.365 [a]	6.944 [a]	6.495 [a]	6.366 [a]
	β_{10}	8.400	7.210 [c]	6.377 [a]	6.561 [a]	7.988 [a]	7.573 [a]	7.035 [a]	7.028 [a]
ARCH	β_2	1.799 [a]	1.609 [a]	1.653 [a]	1.485 [a]	1.852 [a]	1.628 [a]	1.679 [a]	1.475 [a]
	β_3	2.710 [a]	2.370 [a]	2.462 [a]	2.175 [a]	2.766 [a]	2.406 [a]	2.494 [a]	2.170 [a]
	β_4	3.549 [b]	3.181 [a]	3.252 [a]	2.896 [a]	3.663 [a]	3.186 [a]	3.301 [a]	2.869 [a]
	β_5	4.362 [d]	3.994 [a]	3.892 [a]	3.609 [a]	4.534 [a]	3.961 [a]	4.092 [a]	3.560 [a]
	β_6	5.141	4.758 [a]	4.426 [a]	4.307 [a]	5.335 [a]	4.741 [a]	4.848 [a]	4.254 [a]
	β_7	5.816	5.423 [a]	4.976 [a]	5.020 [a]	6.025 [a]	5.495 [a]	5.444 [a]	4.951 [a]
	β_8	6.643	6.010 [a]	5.411 [a]	5.534 [a]	6.747 [a]	6.254 [a]	5.980 [a]	5.668 [a]
	β_9	7.285	6.538 [b]	5.920 [a]	6.030 [a]	7.409 [a]	7.006 [a]	6.538 [a]	6.394 [a]
	β_{10}	7.076	7.104 [c]	6.365 [a]	6.573 [a]	8.125 [b]	7.704 [a]	7.117 [a]	7.130 [a]
ARMA	β_2	1.379 [a]	1.113 [a]	1.213 [a]	1.019 [a]	1.362 [a]	1.088 [a]	1.191 [a]	0.988 [a]
	β_3	1.830 [a]	1.355 [a]	1.544 [a]	1.208 [a]	1.796 [a]	1.311 [a]	1.504 [a]	1.158 [a]
	β_4	2.268 [a]	1.592 [a]	1.866 [a]	1.394 [a]	2.233 [a]	1.539 [a]	1.820 [a]	1.329 [a]
	β_5	2.704 [a]	1.830 [a]	2.188 [a]	1.581 [a]	2.673 [a]	1.771 [a]	2.138 [a]	1.502 [a]
	β_6	3.115 [a]	2.070 [a]	2.497 [a]	1.770 [a]	3.104 [a]	2.005 [a]	2.452 [a]	1.677 [a]
	β_7	3.570 [a]	2.301 [a]	2.828 [a]	1.960 [a]	3.530 [a]	2.239 [a]	2.764 [a]	1.853 [a]
	β_8	3.952 [a]	2.528 [a]	3.072 [a]	2.146 [a]	3.949 [a]	2.472 [a]	3.072 [a]	2.030 [a]
	β_9	4.223 [a]	2.743 [a]	3.288 [a]	2.326 [a]	4.362 [a]	2.702 [a]	3.376 [a]	2.205 [a]
	β_{10}	4.332 [a]	2.940 [a]	3.358 [a]	2.501 [a]	4.783 [a]	2.930 [a]	3.684 [a]	2.379 [a]

Note: Superscript denotes significance at levels of (a) 1%, (b) 2%, (c) 5% and (d) 10%.

Table 6
Empirical power of Kočenda's test against five processes based on Barnett (1997)

A. The sample size T = 500

Process	ε-range	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
FEIG	(0.25σ-1.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
	(0.50σ-1.50σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
	(0.25σ-2.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
GARCH	(0.25σ-1.00σ)	11.1%	10.0%	7.0%	6.0%	7.4%	6.6%	5.1%	2.7%	1.4%
	(0.50σ-1.50σ)	28.4%	38.3%	43.2%	41.1%	33.3%	25.9%	26.5%	26.0%	25.5%
	(0.25σ-2.00σ)	28.5%	35.3%	31.4%	36.7%	43.6%	47.9%	49.5%	51.6%	52.1%
NLMA	(0.25σ-1.00σ)	28.1%	28.8%	17.0%	15.7%	16.0%	13.4%	9.4%	3.7%	0.4%
	(0.50σ-1.50σ)	73.2%	90.2%	93.8%	91.8%	83.7%	70.5%	67.8%	62.2%	56.0%
	(0.25σ-2.00σ)	71.2%	85.1%	80.7%	88.4%	91.9%	91.9%	91.3%	90.2%	88.9%
ARCH	(0.25σ-1.00σ)	76.7%	60.4%	30.8%	19.2%	16.9%	12.7%	7.1%	3.1%	0.3%
	(0.50σ-1.50σ)	97.6%	97.8%	97.1%	93.4%	85.9%	68.8%	67.2%	64.0%	57.1%
	(0.25σ-2.00σ)	97.1%	96.7%	90.1%	89.0%	91.6%	91.4%	90.1%	88.9%	86.9%
ARMA	(0.25σ-1.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	99.9%
	(0.50σ-1.50σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
	(0.25σ-2.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%

Note: The entries are rejection rates in %, computed at the 5% level.

B. The sample size T = 1000

Process	ε-range	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
FEIG	(0.25σ-1.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
	(0.50σ-1.50σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
	(0.25σ-2.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
GARCH	(0.25σ-1.00σ)	24.3%	21.8%	12.3%	7.1%	6.5%	8.1%	6.7%	5.8%	4.1%
	(0.50σ-1.50σ)	49.5%	65.4%	71.5%	70.4%	66.9%	53.5%	40.4%	43.2%	42.6%
	(0.25σ-2.00σ)	50.5%	60.6%	57.9%	46.2%	60.6%	66.7%	71.4%	74.8%	75.4%
NLMA	(0.25σ-1.00σ)	70.0%	76.9%	57.0%	23.1%	27.9%	28.5%	26.8%	18.6%	10.5%
	(0.50σ-1.50σ)	95.3%	99.8%	100.0%	100.0%	99.9%	99.0%	93.4%	92.9%	90.2%
	(0.25σ-2.00σ)	95.9%	99.4%	99.2%	97.6%	98.9%	99.8%	99.9%	99.9%	100.0%
ARCH	(0.25σ-1.00σ)	99.2%	96.3%	79.5%	38.2%	29.7%	26.6%	21.4%	12.4%	6.8%
	(0.50σ-1.50σ)	100.0%	100.0%	100.0%	100.0%	99.8%	98.0%	89.8%	87.8%	86.3%
	(0.25σ-2.00σ)	100.0%	100.0%	100.0%	98.7%	99.1%	99.3%	99.6%	99.4%	98.7%
ARMA	(0.25σ-1.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
	(0.50σ-1.50σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
	(0.25σ-2.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%

Note: The entries are rejection rates in %, computed at the 5% level.

C. The sample size T = 2500

Process	ε-range	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
FEIG	(0.25σ-1.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
	(0.50σ-1.50σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
	(0.25σ-2.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
GARCH	(0.25σ-1.00σ)	65.4%	68.5%	53.8%	29.1%	11.7%	14.6%	15.0%	14.1%	11.6%
	(0.50σ-1.50σ)	86.7%	96.6%	98.5%	98.8%	98.3%	97.5%	92.3%	76.7%	75.3%
	(0.25σ-2.00σ)	86.6%	95.6%	96.9%	92.1%	84.3%	91.4%	94.9%	95.7%	96.9%
NLMA	(0.25σ-1.00σ)	99.9%	100.0%	99.7%	94.6%	51.7%	60.9%	57.7%	55.1%	42.7%
	(0.50σ-1.50σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	99.9%
	(0.25σ-2.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
ARCH	(0.25σ-1.00σ)	100.0%	100.0%	100.0%	98.1%	62.7%	53.8%	47.3%	41.0%	30.8%
	(0.50σ-1.50σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	99.7%	99.1%
	(0.25σ-2.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
ARMA	(0.25σ-1.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
	(0.50σ-1.50σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
	(0.25σ-2.00σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%

Note: The entries are rejection rates in %, computed at the 5% level.

Table 7

Quantiles of the slope coefficients β_m for ε-range (0.60σ – 1.90σ)

A. Quantiles of the slope coefficients β_m for a sample size of 500 observations

Quantile	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
0.5%	1.544	2.305	3.056	3.785	4.505	5.197	5.771	6.254	6.710
1.0%	1.552	2.319	3.075	3.821	4.546	5.250	5.823	6.322	6.795
2.5%	1.564	2.339	3.106	3.862	4.605	5.323	5.898	6.417	6.909
5.0%	1.573	2.353	3.128	3.895	4.646	5.378	5.960	6.498	6.997
95.0%	1.645	2.475	3.312	4.161	5.033	5.846	6.526	7.187	7.835
97.5%	1.650	2.483	3.326	4.183	5.068	5.886	6.576	7.248	7.912
99.0%	1.654	2.492	3.341	4.207	5.106	5.931	6.628	7.310	7.991
99.5%	1.657	2.497	3.350	4.224	5.132	5.967	6.662	7.359	8.057

Note: "*m*" denotes an embedding dimension. Based on 20,000 replications.

B. Quantiles of the slope coefficients β_m for a sample size of 1000 observations

Quantile	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
0.5%	1.566	2.342	3.113	3.874	4.628	5.366	6.078	6.700	7.217
1.0%	1.571	2.351	3.124	3.892	4.652	5.395	6.115	6.746	7.271
2.5%	1.578	2.362	3.143	3.917	4.685	5.436	6.173	6.804	7.349
5.0%	1.584	2.373	3.158	3.938	4.710	5.475	6.221	6.855	7.412
95.0%	1.634	2.456	3.281	4.112	4.951	5.806	6.660	7.331	7.985
97.5%	1.638	2.462	3.291	4.124	4.970	5.835	6.691	7.374	8.035
99.0%	1.642	2.468	3.299	4.140	4.992	5.871	6.731	7.420	8.093
99.5%	1.645	2.472	3.305	4.149	5.006	5.891	6.759	7.453	8.133

Note: "*m*" denotes an embedding dimension. Based on 20,000 replications.

C. Quantiles of the slope coefficients β_m for a sample size of 2500 observations

Quantile	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
0.5%	1.584	2.373	3.158	3.938	4.715	5.486	6.242	6.982	7.697
1.0%	1.587	2.377	3.164	3.948	4.727	5.502	6.265	7.011	7.730
2.5%	1.591	2.384	3.174	3.962	4.746	5.525	6.294	7.054	7.784
5.0%	1.595	2.390	3.183	3.974	4.762	5.546	6.323	7.087	7.825
95.0%	1.626	2.441	3.258	4.078	4.899	5.727	6.563	7.418	8.191
97.5%	1.628	2.445	3.264	4.085	4.911	5.741	6.586	7.449	8.225
99.0%	1.631	2.449	3.270	4.094	4.925	5.759	6.612	7.486	8.266
99.5%	1.633	2.453	3.275	4.101	4.932	5.770	6.626	7.513	8.291

Note: "*m*" denotes an embedding dimension. Based on 20,000 replications.

Table 8
Sensitivity of critical values with respect to ε-range choice

A. The sample size T = 500

ε-ranges	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
(0.60σ-1.90σ) vs (0.25σ-1.00σ)	0.301	0.428	0.504	0.491	0.431	0.306	0.194	-0.063	-0.724
(0.60σ-1.90σ) vs (0.25σ-1.00σ)	0.283	0.453	0.661	0.774	0.897	1.131	1.626	2.412	4.085
(0.60σ-1.90σ) vs (0.50σ-1.50σ)	0.132	0.195	0.254	0.301	0.326	0.258	0.266	0.300	0.319
(0.60σ-1.90σ) vs (0.50σ-1.50σ)	0.120	0.184	0.255	0.342	0.399	0.388	0.499	0.630	0.787
(0.60σ-1.90σ) vs (0.25σ-2.00σ)	0.170	0.249	0.306	0.224	0.096	-0.052	-0.098	-0.115	-0.128
(0.60σ-1.90σ) vs (0.25σ-2.00σ)	0.148	0.232	0.303	0.198	0.027	-0.104	-0.132	-0.168	-0.198

B. The sample size T = 1000

ε-ranges	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
(0.60σ-1.90σ) vs (0.25σ-1.00σ)	0.304	0.446	0.558	0.604	0.594	0.548	0.447	0.345	0.230
(0.60σ-1.90σ) vs (0.25σ-1.00σ)	0.280	0.433	0.620	0.798	0.900	0.999	1.158	1.543	2.113
(0.60σ-1.90σ) vs (0.50σ-1.50σ)	0.131	0.197	0.259	0.316	0.362	0.385	0.328	0.316	0.360
(0.60σ-1.90σ) vs (0.50σ-1.50σ)	0.121	0.182	0.246	0.319	0.408	0.479	0.416	0.519	0.637
(0.60σ-1.90σ) vs (0.25σ-2.00σ)	0.168	0.250	0.322	0.339	0.239	0.110	-0.038	-0.109	-0.130
(0.60σ-1.90σ) vs (0.25σ-2.00σ)	0.149	0.227	0.318	0.341	0.230	0.068	-0.113	-0.136	-0.160

C. The sample size T = 2500

ε-ranges	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
(0.60σ-1.90σ) vs (0.25σ-1.00σ)	0.301	0.448	0.585	0.694	0.737	0.744	0.712	0.644	0.547
(0.60σ-1.90σ) vs (0.25σ-1.00σ)	0.282	0.426	0.582	0.771	0.936	1.038	1.125	1.217	1.432
(0.60σ-1.90σ) vs (0.50σ-1.50σ)	0.130	0.194	0.259	0.321	0.381	0.433	0.469	0.444	0.372
(0.60σ-1.90σ) vs (0.50σ-1.50σ)	0.122	0.183	0.245	0.309	0.379	0.459	0.565	0.525	0.517
(0.60σ-1.90σ) vs (0.25σ-2.00σ)	0.164	0.246	0.326	0.394	0.398	0.301	0.181	0.031	-0.116
(0.60σ-1.90σ) vs (0.25σ-2.00σ)	0.152	0.228	0.308	0.402	0.411	0.311	0.170	-0.026	-0.135

Note: The entries are differences between critical values of two different ranges; a reference range is the range (0.60σ-1.90σ) and a second range is one of the ranges (0.25σ-1.00σ), (0.50σ-1.50σ), or (0.25σ-2.00σ); for each interval the differences are computed for 2.5% quantile (the first row) and 97.5% quantile (the second row).

Table 9
Empirical power of Kočenda's test against five processes based on Barnett et al. (1997)

A. The sample size T = 500

Process	ε-range	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
FEIG	(0.60σ-1.90σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
GARCH	(0.60σ-1.90σ)	32.3%	46.9%	52.6%	56.5%	56.0%	52.9%	44.9%	46.6%	48.1%
NLMA	(0.60σ-1.90σ)	80.3%	94.5%	97.7%	97.7%	97.4%	94.9%	88.6%	87.0%	85.7%
ARCH	(0.60σ-1.90σ)	98.4%	99.0%	98.8%	98.3%	97.3%	94.1%	88.8%	86.4%	83.3%
ARMA	(0.60σ-1.90σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%

Note: The entries are rejection rates in %, computed at the 5% level.

B. The sample size T = 1000

Process	ε-range	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
FEIG	(0.60σ-1.90σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
GARCH	(0.60σ-1.90σ)	55.4%	71.4%	78.7%	83.6%	85.4%	83.3%	79.3%	70.5%	70.6%
NLMA	(0.60σ-1.90σ)	97.4%	99.8%	100.0%	100.0%	100.0%	100.0%	100.0%	99.9%	99.9%
ARCH	(0.60σ-1.90σ)	100.0%	100.0%	100.0%	100.0%	100.0%	99.9%	99.8%	98.7%	98.3%
ARMA	(0.60σ-1.90σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%

Note: The entries are rejection rates in %, computed at the 5% level.

C. The sample size T = 2500

Process	ε-range	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
FEIG	(0.60σ-1.90σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
GARCH	(0.60σ-1.90σ)	88.2%	97.7%	99.3%	99.5%	99.7%	99.8%	99.7%	99.0%	98.0%
NLMA	(0.60σ-1.90σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
ARCH	(0.60σ-1.90σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
ARMA	(0.60σ-1.90σ)	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%

Note: The entries are rejection rates in %, computed at the 5% level.

Table 10
Computed Kočenda test statistics for residuals from Kugler and Lenz (1990)

Currency	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
CHF	1.541 [a]	2.312 [b]	3.094 [c]	3.862 [d]	4.626 [d]	5.462	6.139	6.618	7.243
DEM	1.501 [a]	2.242 [a]	2.948 [a]	3.607 [a]	4.259 [a]	4.924 [a]	5.544 [a]	5.937 [a]	6.562 [a]
FRF	1.476 [a]	2.198 [a]	2.868 [a]	3.501 [a]	4.141 [a]	4.788 [a]	5.407 [a]	5.863 [a]	6.345 [a]
JPY	1.421 [a]	2.087 [a]	2.730 [a]	3.371 [a]	4.002 [a]	4.656 [a]	5.290 [a]	5.892 [a]	6.338 [a]

Note: Superscript denotes significance at levels of (a) 1%, (b) 2%, (c) 5% and (d) 10%.

Table 11
Computed Kočenda test statistics for residuals from Brock, Hsieh, and LeBaron (1993)

Currency	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
CAD	1.506 [a]	2.253 [a]	2.997 [a]	3.744 [a]	4.487 [a]	5.221 [a]	5.938 [a]	6.685 [a]	7.387 [a]
CHF	1.457 [a]	2.178 [a]	2.896 [a]	3.614 [a]	4.335 [a]	5.061 [a]	5.768 [a]	6.493 [a]	7.177 [a]
DEM	1.466 [a]	2.198 [a]	2.918 [a]	3.613 [a]	4.254 [a]	4.875 [a]	5.481 [a]	6.038 [a]	6.546 [a]
GBP	1.293 [a]	1.886 [a]	2.451 [a]	2.945 [a]	3.366 [a]	3.720 [a]	3.986 [a]	4.188 [a]	4.354 [a]

Note: Superscript denotes significance at levels of (a) 1%, (b) 2%, (c) 5% and (d) 10%.

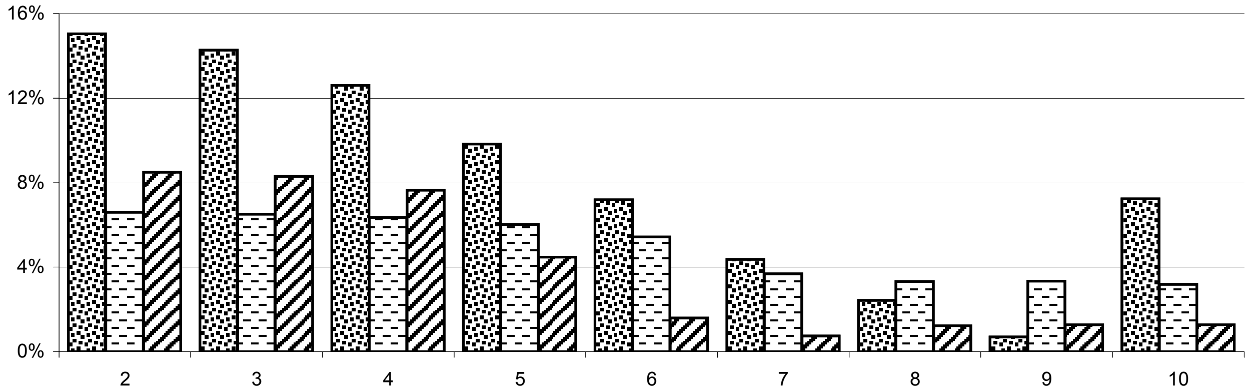
Table 12
Computed Kočenda test statistics for residuals from Kugler and Lenz (1993)

Currency	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
AUD	1.238 [a]	1.810 [a]	2.368 [a]	2.916 [a]	3.452 [a]	3.978 [a]	4.499 [a]	5.034 [a]	5.550 [a]
BEF	1.468 [a]	2.201 [a]	2.904 [a]	3.586 [a]	4.272 [a]	4.961 [a]	5.680 [a]	6.284 [b]	6.775 [b]
CAD	1.451 [a]	2.171 [a]	2.857 [a]	3.477 [a]	4.071 [a]	4.635 [a]	5.158 [a]	5.594 [a]	5.830 [a]
CHF	1.527 [a]	2.296 [a]	3.066 [b]	3.819 [b]	4.553 [c]	5.349 [d]	6.114	6.581	7.170
DEM	1.497 [a]	2.248 [a]	2.943 [a]	3.607 [a]	4.265 [a]	4.959 [a]	5.602 [a]	6.089 [a]	6.600 [a]
ESP	1.377 [a]	2.027 [a]	2.640 [a]	3.221 [a]	3.690 [a]	4.080 [a]	4.388 [a]	4.581 [a]	4.786 [a]
FRF	1.494 [a]	2.247 [a]	2.952 [a]	3.659 [a]	4.359 [a]	5.083 [a]	5.719 [a]	6.158 [a]	6.691 [a]
ITL	1.497 [a]	2.248 [a]	2.956 [a]	3.635 [a]	4.270 [a]	4.893 [a]	5.438 [a]	5.881 [a]	6.444 [a]
JPY	1.443 [a]	2.145 [a]	2.847 [a]	3.540 [a]	4.227 [a]	4.877 [a]	5.573 [a]	6.184 [a]	6.647 [a]
NLG	1.512 [a]	2.272 [a]	2.996 [a]	3.716 [a]	4.449 [a]	5.221 [b]	5.891 [c]	6.436 [d]	6.956 [d]

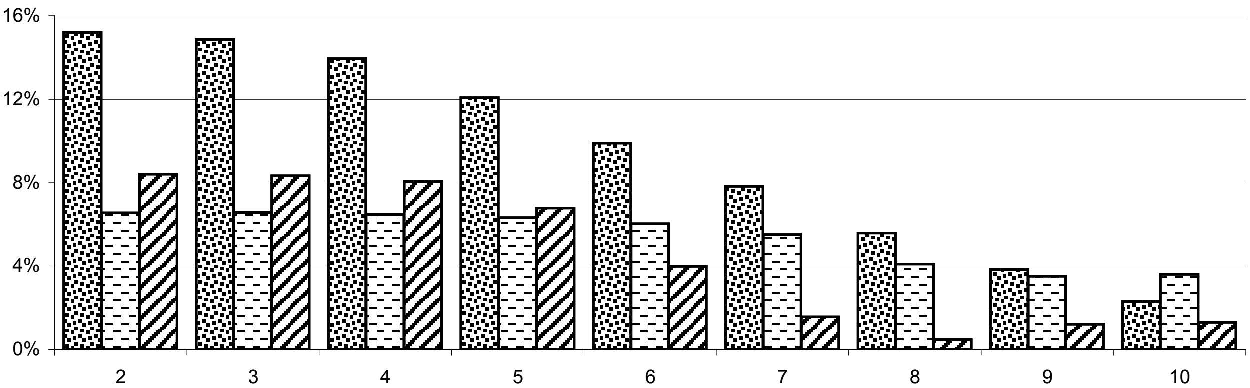
Note: Superscript denotes significance at levels of (a) 1%, (b) 2%, (c) 5% and (d) 10%.

Figure 1a
Relative differences of critical values with respect to ε-range choice
(in absolute value)

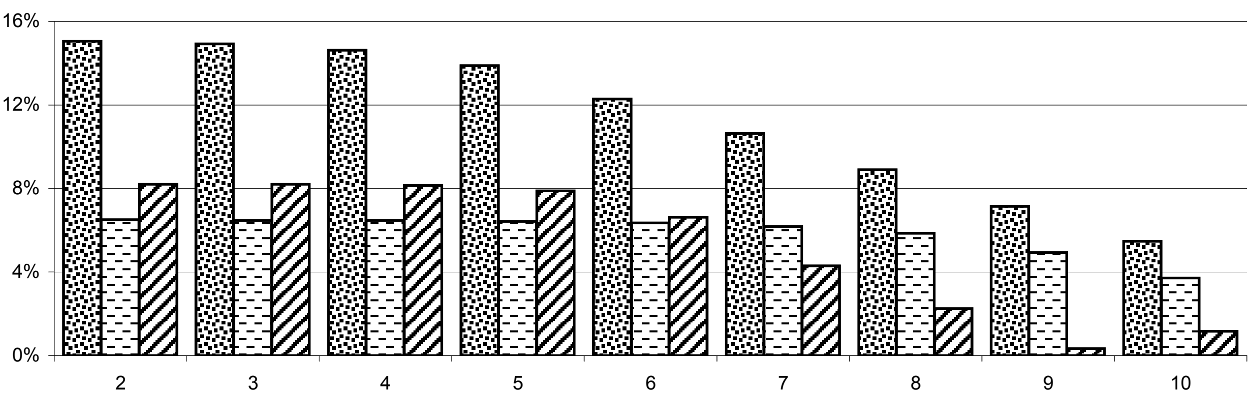
A. Dispersion of 2.5% quantile for 500 observations



B. Dispersion of 2.5% quantile for 1000 observations



C. Dispersion of 2.5% quantile for 2500 observations

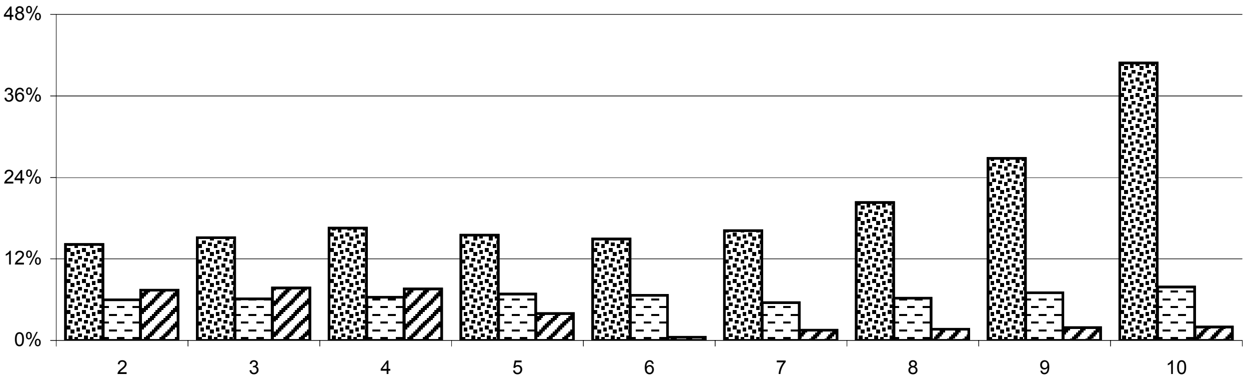


Note: The entries are relative changes of critical values; they are computed as appropriate quantile differences between two intervals over embedding dimension *m;* a reference range is the range (0.60σ-1.90σ) and a second range is one of the ranges (0.25σ-1.00σ), (0.50σ-1.50σ), or (0.25σ-2.00σ). For better understanding we picture them in absolute values.

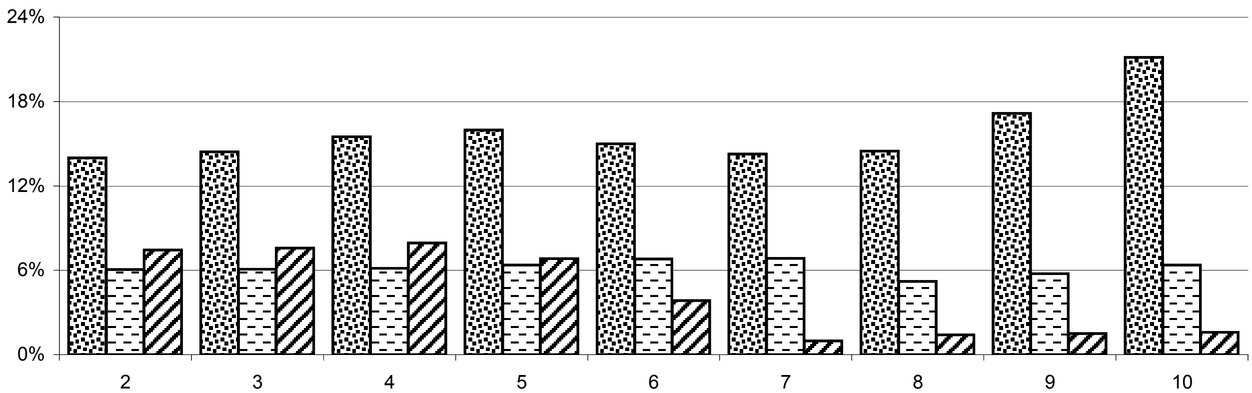
(0.60σ-1.90σ) vs (0.25σ-1.00σ) (0.60σ-1.90σ) vs (0.50σ-1.50σ) (0.60σ-1.90σ) vs (0.25σ-2.00σ)

Figure 1b
Relative differences of critical values with respect to ε-range choice
(in absolute value)

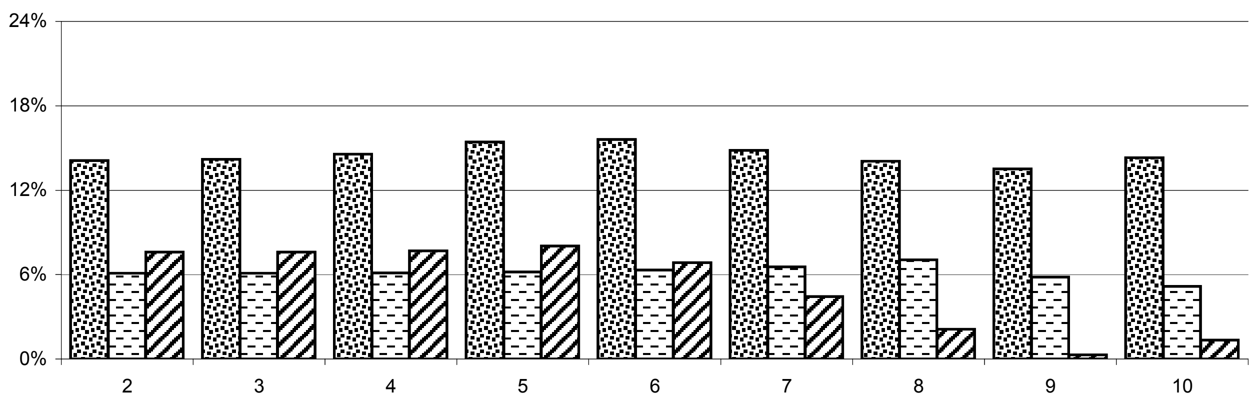
A. Dispersion of 97.5% quantile for 500 observations



B. Dispersion of 97.5% quantile for 1000 observations



C. Dispersion of 97.5% quantile for 2500 observations



Note: The entries are relative changes of critical values; they are computed as appropriate quantile differences between two intervals over embedding dimension *m;* a reference range is the range (0.60σ-1.90σ) and a second range is one of the ranges (0.25σ-1.00σ), (0.50σ-1.50σ), or (0.25σ-2.00σ). For better understanding we picture them in absolute values.